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Randomly Transitional Phenomena in the System Governed by Duffing's Equation

Yoshisuke Ueda[1]

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This paper deals with turbulent or chaotic phenomena which occur in the system governed by Duffing's equation, a special type of two-dimensional periodic system. By using analog and digital computers, experiments are carried out with special reference to the change of attractors and of average power spectra of the random processes under the variation of the system parameters. On the basis of the experimental results, an outline of the random process is made clear. The results obtained in this paper will be applied to various physical problems and will also serve as material for the development of a proper mathematics of this phenomenon.

KEY WORDS: Randomly transitional phenomenon; Duffing's equation; C^∞-diffeomorphism; computer simulation; global structure of solutions of nonlinear differential equation; bundle of solutions; attractor; branching; catastrophe; extinction.

1. INTRODUCTION

Various physical problems are described by Duffing's equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + f(x) = e(t) \tag{1}$$

where $e(t)$ is a periodic function of the period 2π. This equation itself is simple and deterministic, but in the real system governed by Eq. (1) certain random phenomena can be observed. Such random phenomena could be attributed to small, uncertain factors which are usually neglected in formulating mathematical models from the real systems.

The uncertain factors lie between causes and effects in physical systems. When these factors are small, their influence can be neglected in many cases and the phenomena are treated as deterministic processes. But in

[1] Department of Electrical Engineering, Kyoto University, Kyoto, Japan.

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nonlinear systems, however small uncertain factors may be, they sometimes bring statistical properties into phenomena, depending upon the global structure of the solutions of the differential equations of the system.

Recently, phenomena of this kind, generally called turbulent or chaotic behavior, have been vigorously attacked by many researchers in various fields, such as plasma physics, fluid dynamics, and chemical reactions, as well as in biological, economic and social sciences. The phenomena are treated by difference equations[(1-3)] and also by ordinary differential equations.[(4-7)] Mathematical justifications of the treatments have been argued by many researchers. Of these, the work by Ruelle and Takens,[(8)] Bowen,[(9)] and Marsden *et al.*[(10,11)] is notable.

We have carried out computer experiments for the electric circuit with nonlinear inductance under the impression of a sinusoidal voltage. The circuit is described by Duffing's equation. In the computer-simulated system of the equation, turbulent or chaotic oscillatory phenomena are observed for some values of the system parameters. On the basis of the experimental results, we present the following interpretations about the phenomenon.[(12)] The phenomenon cannot be represented by a single solution of the equation, but by a bundle of solutions which is asymptotically (orbitally) stable and contains infinitely many unstable periodic solutions. The representative point of the physical state wanders randomly among the solutions of this bundle under the influence of small uncertain factors, such as random noise. Though the statistical properties arise from uncertain factors, the power spectrum of the random process depends practically not on the nature of uncertain factors but on the structure of the bundle of solutions. We have called the phenomenon represented by the bundle of solutions the randomly transitional phenomenon.[(13)]

This paper also discusses the randomly transitional process in the system governed by Duffing's equation with particular attention to the transition of the process under the variation of the system parameters.

2. RANDOMLY TRANSITIONAL PROCESS

Following the previous report,[(12)] we consider the equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B\cos t \tag{2}$$

or

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -ky - x^3 + B\cos t \tag{3}$$

as a specific example of Duffing's equation.

2.1. Bundle of Solutions Representing Randomly Transitional Process

To begin with, we show illustrative pictures for a system governed by Eqs. (3) with the values of parameters k and B specified by the following equations:

$$\lambda = (k, B) = (0.1, 12.0) \tag{4}$$

In the system two types of steady phenomena occur, depending on the initial conditions, one of which represents a deterministic process and the other a randomly transitional process. Solution curves representing steady states in the txy space are shown in Fig. 1. Figure 1a shows the periodic solution representing the deterministic process. Figure 1b shows the outline of a

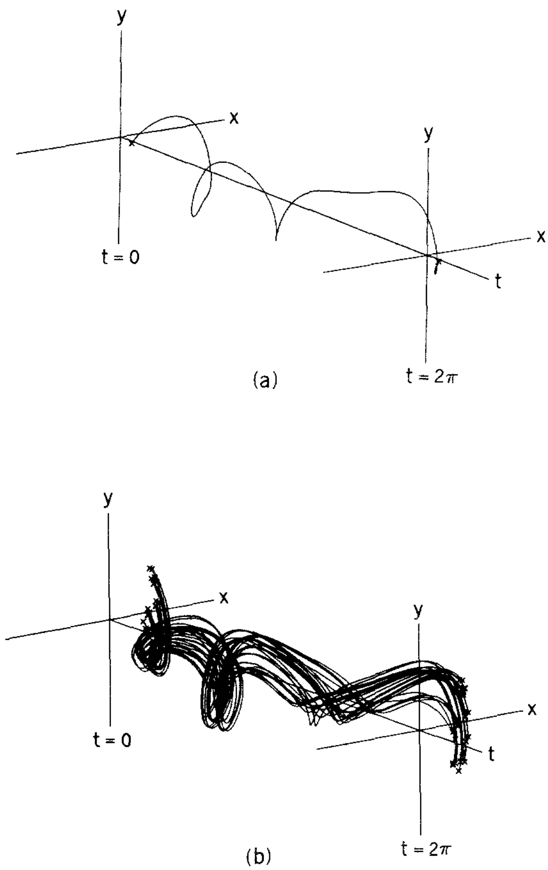


Fig. 1. Solution curves in the txy space.

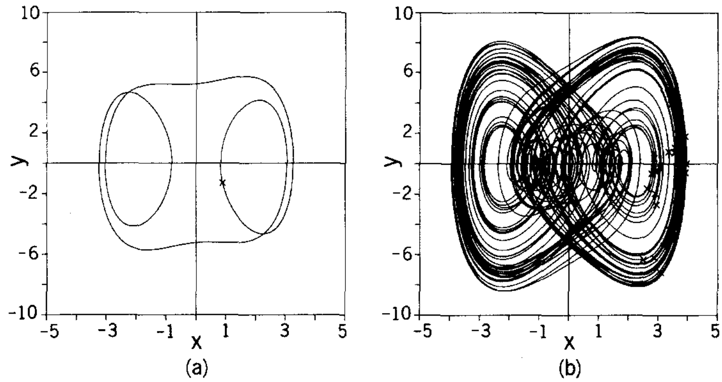


Fig. 2. Trajectories of the computer solution.

bundle of solutions representing the random process. The bundle is asymptotically (orbitally) stable and is reproducible in every computer experiment. Figures 2a and 2b show the trajectories of computer solutions in the xy plane. In the figures, positions of the representative point at the instant $t = 2n\pi$ ($n \in Z^+$) are marked $\times$. The trajectory of Fig. 2b is not reproducible in every analog computer simulation. Therefore, this trajectory is a realization of the randomly transitional process.

Here we introduce a discrete dynamical system on the xy plane by using the solutions of Eqs. (3). To this end, let us consider the solution $x = x(t, x_0, y_0)$, $y = y(t, x_0, y_0)$ of Eqs. (3), which, when $t = 0$, is at the point $p_0 = (x_0, y_0)$ of the xy plane. Let $p_1 = (x_1, y_1)$ denote the point specified by $x_1 = x(2\pi, x_0, y_0)$, $y_1 = y(2\pi, x_0, y_0)$; then we define a C^∞-diffeomorphism f_λ

$$f_\lambda: \quad R^2 \to R^2, \quad p_0 \mapsto p_1, \quad \lambda = (k, B) \tag{5}$$

of the xy plane into itself, or a discrete dynamical system on R^2.

The steady state in the system governed by Eqs. (3) is represented by an attractor of the diffeomorphism f_λ. Figures 3a and 3b show the attractors observed in the same system as in Figs. 1 and 2. Figure 3a shows a completely stable fixed point representing the deterministic periodic process, while Fig. 3b shows an outline of the attractor representing the randomly transitional process.

2.2. Summary of the Previous Investigation[12]

In this section we briefly explain the main results on the randomly transitional process in the system governed by Eqs. (3) obtained in our previous investigation.

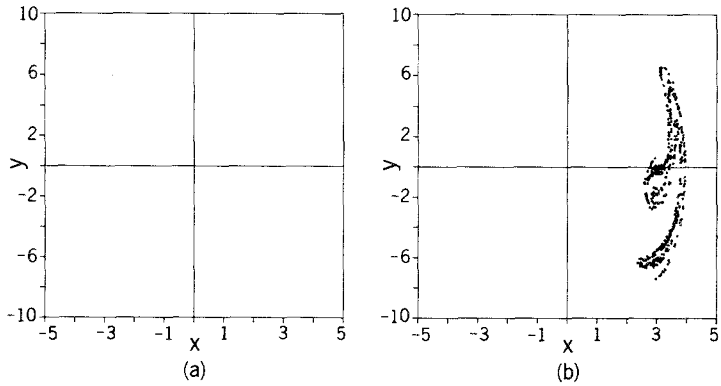


Fig. 3. Outlines of the attractors.

(1) A randomly transitional process occurs resulting from the global structure of the solutions of the nonlinear differential equation. This phenomenon is not a special one which appears only for particular values of the system parameters, but can be observed in a rather wide range of values.

(2) The bundle of solutions is a set of solutions whose initial points at $t = 0$ belong to the attractor of the corresponding diffeomorphism f_λ. The attractor will reasonably be defined by the asymptotically stable, invariant, closed set of f_λ containing infinitely many unstable minimal sets which are connected to one another by the influence of uncertain factors in the real system. The attractor M corresponding to Fig. 3b is identical with a closure of unstable manifolds of the directory unstable fixed point ${}^1D^1$ of f_λ, i.e., $M = \mathrm{Cl}\, W^u({}^1D^1)$. The set $M = \mathrm{Cl}\, W^u({}^1D^1)$ contains infinitely many periodic points. In fact, as shown in Fig. 4 of Ref. 12, the unstable manifolds $W^u({}^1D^1)$ intersect the stable manifolds $W^s({}^1D^1)$ forming a homoclinic cycle. However, the existence of a minimal set representing a nonperiodic solution is not known. The structural stability of M in the sense of Andronov–Pontryagin seems not to hold.

(3) Experimental results reveal that mean values $m_X(t)$ and $m_Y(t)$ of the randomly transitional processes $\{X(t)\}$ and $\{Y(t)\}$ are periodic functions of t with period 2π. It is also clear that the bundle of solutions is periodic in t with the same period. Hence these processes are regarded as periodic nonstationary processes. As mentioned before, the average power spectra of these processes depend practically not on the nature of uncertain factors but on the structure of the bundle of solutions. The other statistical characteristics of the process depend on the nature of uncertain factors as well as on the structure of the bundle of solutions. However, no attempt has been made to relate the transition probability of the process with the nature of uncertain factors.

3. EXPERIMENTAL RESULTS

By making use of analog and digital computers, we show some experimental results concerned with the randomly transitional process that occurs in the computer-simulated systems governed by Eqs. (3). The region in which this type of process occurs is roughly estimated to be in the range $k = 0\text{–}0.3$ and $B = 6\text{–}13$. Within this range, however, there are other regions in which different types of steady states are sustained and they sometimes overlap partly with one another. Under these circumstances, we focus our attention on a transition of the randomly transitional process in the case when the system parameter, either k or B, is varied from the value specified by Eq. (4).

3.1. Dependence of the Attractors on the System Parameters

First we show the change of attractors of the system governed by Eqs. (3), particularly in the case where the parameter B is varied while k is kept constant, $k = 0.1$. Outlines of the attractors in such a case are shown in Figs. 4. Figure 4a shows a completely stable 3-periodic group of the diffeomorphism f_λ. These periodic points represent an ultrasubharmonic oscillation of order 7/3, i.e., the oscillation whose principal frequency is 7/3 times the frequency of the external force. When B is slightly increased, a fluctuation is brought into the process. This state is shown in Fig. 4b. Further increase in B results in the abrupt growth of the fluctuation and the randomly transitional process develops. The randomly transitional process continues until B reaches 13.3 and the attractors in such a case are shown in Figs. 4c–4g. Needless to say, the representative points of the state which give the outline of the attractor are plotted after the transient has vanished in the computer-simulated system. A catastrophe occurs at some value of B between 13.3 and 13.4, and the randomly transitional process is replaced by the harmonic oscillation. This transient state is sketched in Fig. 4h. When B is decreased from 13.4, the harmonic oscillation is sustained until B reaches 11.6, but below 11.5 only a randomly transitional process occurs. Hence, for values of B between 11.6 and 13.3, there are two types of steady phenomena, i.e., the randomly transitional processes and the harmonic oscillations. An example of boundaries of the domains of attraction for these two states is given in Fig. 5 of Ref. 12.

Next we consider the change of attractors in the case where the parameter k is varied while B is kept constant, $B = 12.0$. Outlines of the attractors in such a case are shown in Fig. 5. From these results, we see that when the damping coefficient k is small, the size of the attractor is large, but as k increases, it decreases, and finally the attractor becomes 2-periodic group and then the fixed point of the diffeomorphism f_λ.

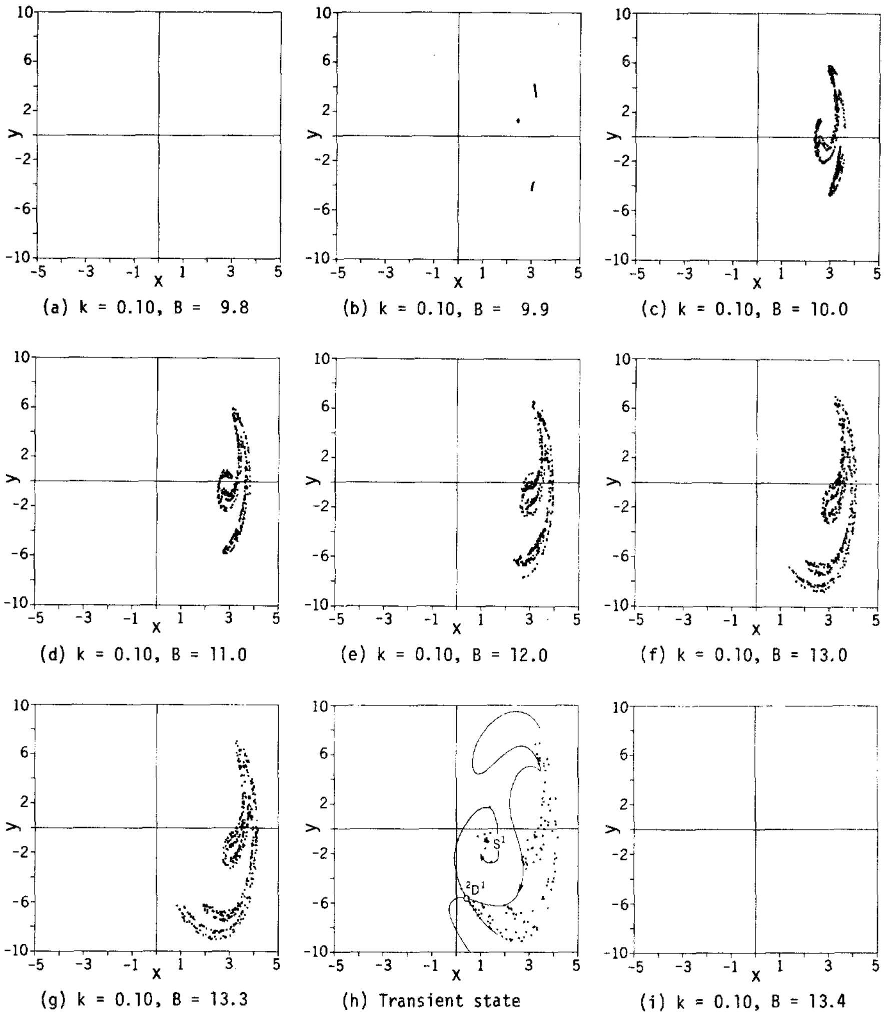


Fig. 4. Outlines of the attractors for various values of B; $k = 0.10$.

3.2. Dependence of the Exponentlike Quantities on the System Parameters

Here, we estimate the exponentlike quantities e_1 and e_2 in order to give the stochasticity properties of the attractors obtained in the preceding section. These quantities are introduced following the characteristic exponents of the fixed (periodic) point. One of them e_1 indicates the rate of divergence of nearby points in the attractor and the other e_2 the rate of attraction of the attractor.

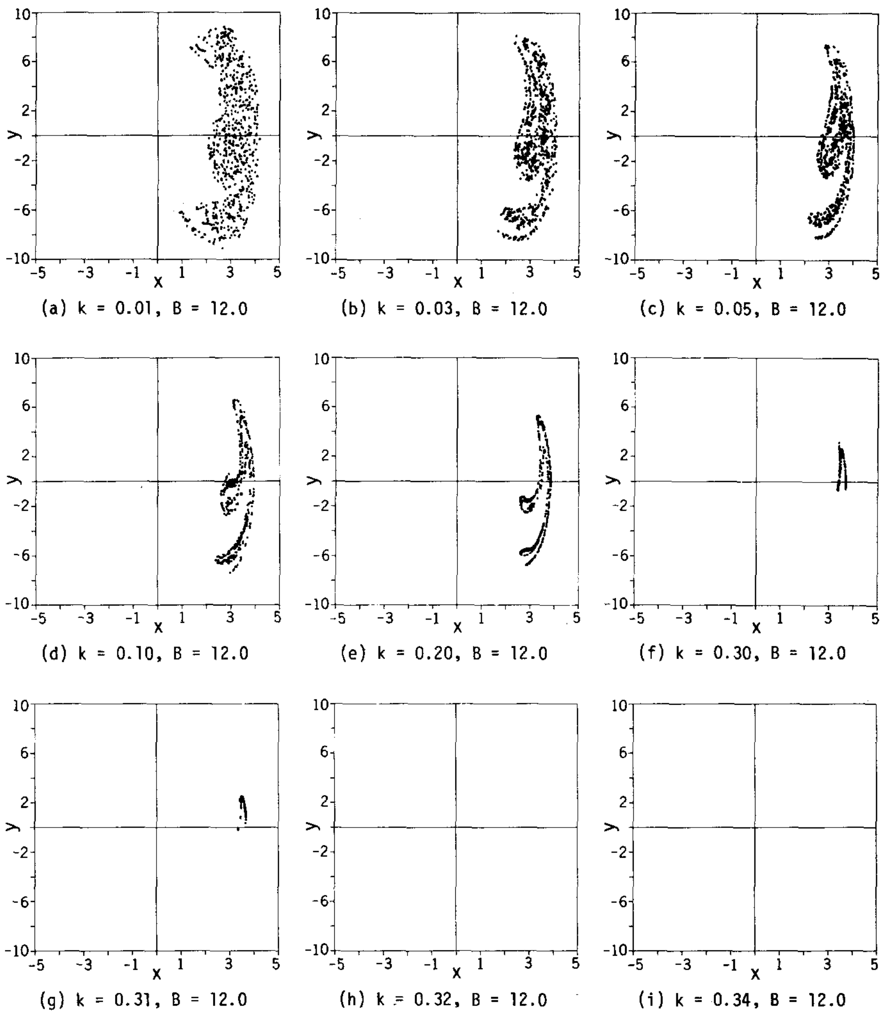


Fig. 5. Outlines of the attractors for various values of k; $B = 12.0$.

We compute the coefficients A', B', C', and D' for sufficiently small d and arbitrary point p_0 by letting $p_1 = f_\lambda(p_0) = (x_1, y_1)$, $q_1 = f_\lambda(q_0) = (x_1 + A'd, y_1 + B'd)$, and $r_1 = f_\lambda(r_0) = (x_1 + C'd, y_1 + D'd)$, where $p_0 = (x_0, y_0)$, $q_0 = (x_0 + d, y_0)$, and $r_0 = (x_0, y_0 + d)$. Then we obtain the quantities $\rho_1 = \max\{|m_1|, |m_2|\}$ and $p_2 = \min\{|m_1|, |m_2|\}$, where m_1 and m_2 are the roots of the following quadratic equation:

$$m^2 - (A' + D')m + (A'D' - B'C') = 0$$

Table I. Exponentlike Quantities Indicating Stochasticity Properties for the Attractors in Figs. 4 and 5[a]

Case	λ: k	λ: B	e_1	e_2
Fig. 4a	0.10	9.8	$-0.050 \pm 0.108i$*,[a]	
b	0.10	9.9	0.065	−0.166
c	0.10	10.0	0.102	−0.202
d	0.10	11.0	0.114	−0.214
e	0.10	12.0	0.149	−0.249
f	0.10	13.0	0.182	−0.282
g	0.10	13.3	0.183	−0.284
h	0.10	13.4	—	
i	0.10	13.4	$-0.050 \pm 0.383i$*	
Fig. 5a	0.01	12.0	0.189	−0.200
b	0.03	12.0	0.179	−0.209
c	0.05	12.0	0.172	−0.222
d	0.10	12.0	0.149	−0.249
e	0.20	12.0	0.149	−0.350
f	0.30	12.0	0.119	−0.419
g	0.31	12.0	0.085	−0.395
h	0.32	12.0	−0.005*	−0.315*
i	0.34	12.0	−0.052*	−0.288*

[a] Asterisks denote characteristic exponents for the periodic solutions.

Proceeding in this manner, we get sequences of positive numbers $\rho_1^{(i)}$ and $\rho_2^{(i)}$ for every image $p_i = f_\lambda{}^i(p_0)$ $(i = 0, 1, 2, \ldots)$ and we can define the exponentlike quantities

$$e_1^{(n)} = \frac{1}{2n\pi} \sum_{i=0}^{n-1} \ln \rho_1^{(i)}, \qquad e_2^{(n)} = \frac{1}{2n\pi} \sum_{i=1}^{n-1} \ln \rho_2^{(i)} \tag{6}$$

From the numerical experiments, we notice that the limits

$$e_1 = \lim_{n\to\infty} e_1^{(n)}, \qquad e_2 = \lim_{n\to\infty} e_2^{(n)} \tag{7}$$

seem to exist and they are independent of both d and p_0. Table I indicates the exponentlike quantities thus obtained for the attractors in Figs. 4 and 5.

3.3. Dependence of the Average Power Spectra on the System Parameters

In this section, we estimate the mean value and the average power spectrum of the randomly transitional process $\{X(t)\}$ for some representative values of the system parameters. To this end, we regard the process $\{X(t)\}$ as the periodic random process $\{X_T(t)\}$ with sufficiently long period T, which is a multiple of 2π. Then a realization $x_T(t)$ is expanded into Fourier series as

$$x_T(t) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_m \cos m\omega_0 t + b_m \sin m\omega_0 t), \qquad \omega_0 = 2\pi/T \tag{8}$$

where

$$a_m = (2/T)\int_{-T/2}^{T/2} x_T(t) \cos m\omega_0 t \, dt, \qquad m = 0, 1, 2, \ldots$$

$$b_m = (2/T)\int_{-T/2}^{T/2} x_T(t) \sin m\omega_0 t \, dt, \qquad m = 1, 2, 3, \ldots$$

From these coefficients, the mean value $m_X(t)$ and the average power spectrum $\Phi_X(\omega)$ of the random process $\{X(t)\}$ are estimated as follows:

$$\begin{aligned} m_X(t) &= \langle X(t)\rangle \fallingdotseq \langle X_T(t)\rangle \\ &= \langle \tfrac{1}{2}a_0\rangle + \sum_{m=1}^{\infty} \{\langle a_m\rangle \cos m\omega_0 t + \langle b_m\rangle \sin m\omega_0 t\} \end{aligned} \tag{9}$$

$$\begin{aligned} \Phi_X(\omega) &= \lim_{T\to\infty} \left\langle (1/T)\left|\int_{-T/2}^{T/2} x_T(t)e^{-i\omega t}\, dt\right|^2\right\rangle \fallingdotseq \Phi_X(m\omega_0) \\ &= (2\pi/\omega_0)\langle \tfrac{1}{4}(a_m{}^2 + b_m{}^2)\rangle, \qquad \omega_0 = 2\pi/T \end{aligned} \tag{10}$$

Let us give the results estimated by using Eqs. (9) and (10) for some selected values of the system parameters in the preceding section. Every mean value $m_X(t)$ turns out to be a periodic function in t with period 2π, having the form

$$m_X(t) = \sum_{m=1,3,5,\cdots}^{\infty} (a_m \cos mt + b_m \sin mt) \tag{11}$$

The parameters used and some of the Fourier coefficients in Eq. (11) are listed in Table II. Figure 6 shows the average power spectra for the same

Table II. Mean Value, Eq. (11), of the Randomly Transitional Process $\{X(t)\}$

	λ		Coefficients in Eq. (11)					
Case	k	B	a_1	b_1	a_3	b_3	a_5	b_5
a	0.10	10.0	1.85	0.17	0.90	−0.01	0.20	0.01
b	0.10	11.0	1.79	0.19	1.09	−0.13	0.24	−0.02
c	0.10	12.0	1.72	0.22	1.21	−0.26	0.25	−0.06
d	0.10	13.0	1.61	0.26	1.30	−0.50	0.24	−0.10
e	0.01	12.0	1.64	0.03	1.12	−0.08	0.18	−0.01
f	0.20	12.0	1.74	0.43	1.17	−0.43	0.24	−0.08

system parameters as in Table II. In the figure, line spectra indicate the periodic components of the mean values. Numerical values attached to the line spectra represent the powers $\frac{1}{4}(a_m{}^2 + b_m{}^2)$, $m = 1, 3, 5,\ldots$, concentrated on those frequencies. Table III indicates the power

$$\lim_{T\to\infty} (1/T) \int_{-T/2}^{T/2} \langle x_T{}^2(t) \rangle \, dt \tag{12}$$

of the random process $\{X(t)\}$. In the table, powers of the periodic component

$$\tfrac{1}{2} \sum_{m=1,3,5,\cdots}^{\infty} (a_m{}^2 + b_m{}^2) \tag{13}$$

and of the random component

$$(1/2\pi) \int_{-\infty}^{\infty} \Phi_X(\omega)\, d\omega - \tfrac{1}{2} \sum_{m=1,3,5,\cdots}^{\infty} (a_m{}^2 + b_m{}^2) \tag{14}$$

are also given.

Table III. Power of the Random Process $\{X(t)\}$

Case	λ: k	λ: B	Power of the process Eq. (12)	Power of periodic component, Eq. (13)	Power of random component, Eq. (14)
a	0.10	10.0	2.79	2.14	0.64
b	0.10	11.0	2.93	2.25	0.68
c	0.10	12.0	3.08	2.30	0.78
d	0.10	13.0	3.29	2.34	0.95
e	0.01	12.0	3.28	2.00	1.27
f	0.20	12.0	3.11	2.42	0.69

4. DISCUSSION

The structure of the attractor and the character of the average power spectrum are investigated in detail in Section 4 of Ref. 12. Some unanswered problems about the phenomenon are also summarized there. Here we discuss the experimental results obtained in the preceding section, with particular attention to the transition of the phenomena under the variation of the system parameters.

In the following descriptions, the symbol ${}^iS_j{}^n$ indicates the ith completely stable n-periodic point and the subscript j ($j = 1, 2,\ldots, n$) represents

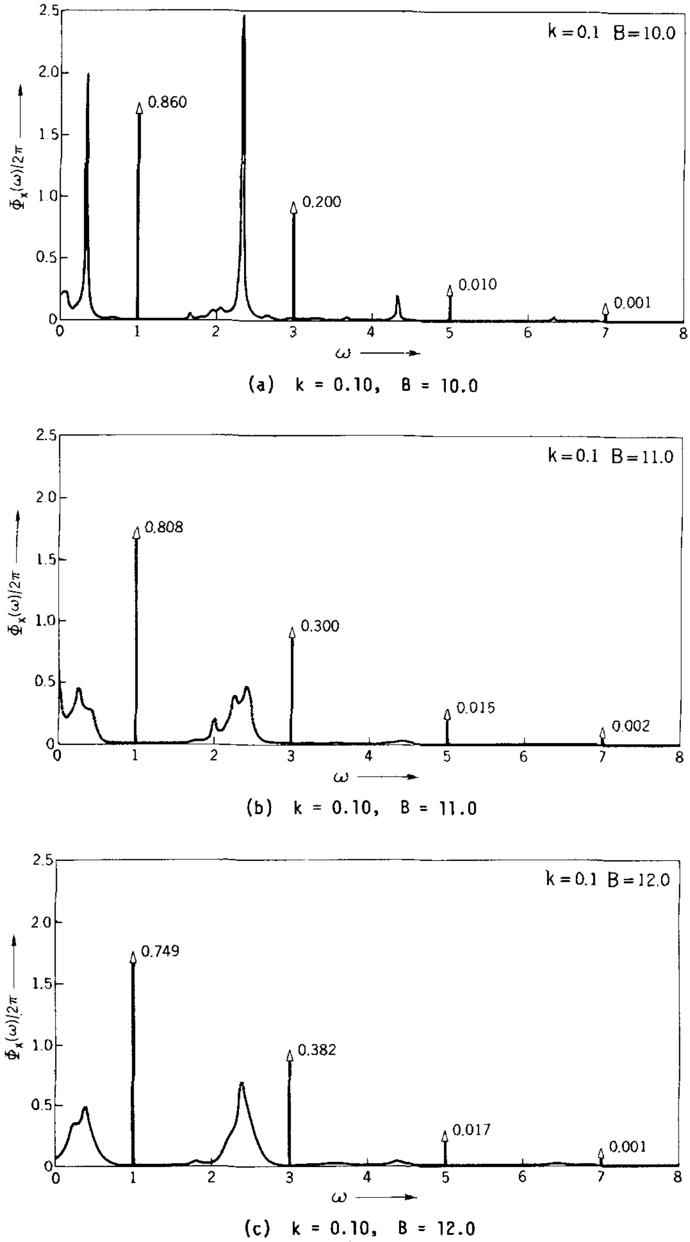
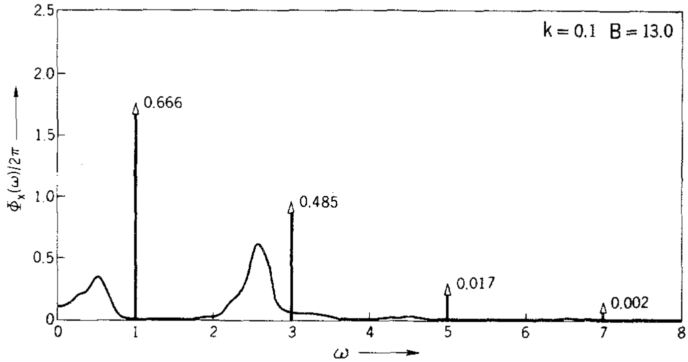
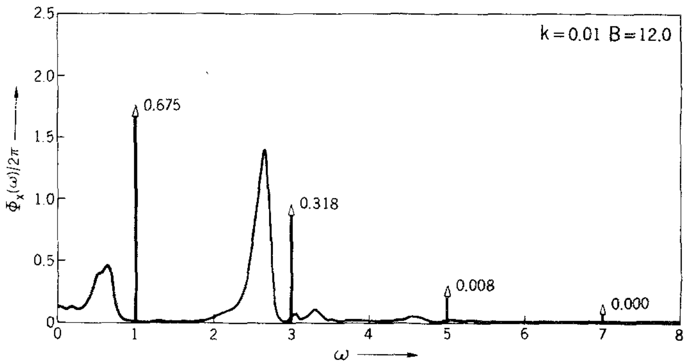


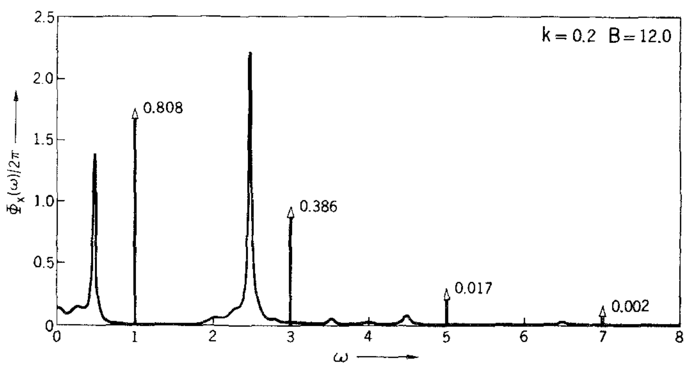
Fig. 6. Average power spectra for various values of $\lambda = (k, B)$.



(d) k = 0.10, B = 13.0



(e) k = 0.01, B = 12.0



(f) k = 0.20, B = 12.0

Fig. 6. Continued.

the order of the successive movements of the images under the diffeomorphism f_λ in the group. Similar symbols are applied to the directly unstable (D) and inversely unstable (I) periodic points. The point S is sometimes called the sink and both D and I are the saddles.

(1) Transition from Fig. 4a to Fig. 4b is regarded as the SI branching; namely,

$$S_1^3 \to (S_1^6 - I_1^3 - S_4^6), \qquad S_1^6 \to (S_1^{12} - I_1^6 - S_7^{12}),\ldots$$

This type of bifurcation is equally called a pitchfork bifurcation. As the branching develops, the domains of attraction for the sinks S_j^n become narrow. When uncertain factors acting on the system become relatively large in comparison with the domains of attraction for the sinks, the randomly transitional phenomenon arises. But the details about the continuation of the order n of SI branching and the appearance of the homoclinic points are not known.

(2) Three point sets in Fig. 4b are regarded as the closure of the unstable manifolds Cl $W^u(I_j^3)$ of the saddles I_j^3 ($j = 1, 2, 3$), which are individually invariant under f_λ^3. As the parameter B increases, these point sets grow rapidly and connect with one another and become the closed invariant set Cl $W^u({}^1D^1)$.

(3) In Fig. 6a, the peaks of the random component exist at $\omega = 1/3$, $7/3$,.... This indicates that the randomly transitional process of this case has developed from the ultrasubharmonic oscillation of order 7/3. We see in Fig. 6b that the peak at $\omega = 7/3$ in Fig. 6a splits into two parts and the peaks appear at $\omega = 2.26$ and 2.41. These are approximately equal to 13/6 and 15/6. This means that the representative point of the state frequently remains in the neighborhood of the solution which passes through the 6-periodic group appeared in the course of SI branching.

We see in Fig. 6 and in Table III that the powers of the random component become large when B is large and k is small. Figures 5e–5h and Fig. 6f show the transition from the randomly transitional process to the ultrasubharmonic oscillation of order 5/2, tracing the inverse process as stated above in (1) and (2).

(4) A catastrophe from Fig. 4g to Fig. 4i occurs when ${}^1D^1$ is chained to ${}^2D^1$ through a transition chain. Though not shown in the figure, the point ${}^1D^1$ is a saddle contained in the attractor of Fig. 4g. A transition chain is made up of branches consisting of the manifolds $W^u({}^1D^1)$ and $W^s({}^2D^1)$ which intersect at heteroclinic points. Outlines of the stable and unstable manifolds of the saddle ${}^2D^1$ are sketched in Fig. 4h. This type of jump phenomenon is first reported here and cannot be accounted for without considering the global aspect of the diffeomorphism f_λ.

(5) When B is decreased, the sink S^1 in Fig. 4h disappears at some

value of B between 11.6 and 11.5. This transition is regarded as SD extinction; namely,

$$S^1\text{–}{}^2D^1 \to \text{coalescence} \to \text{extinction}$$

The jump phenomenon from the harmonic oscillation to the randomly transitional process takes place in this case.

(6) Turbulent or chaotic phenomena which occur in systems governed by three-dimensional autonomous equations, e.g., Lorenz' equations, are well studied and strange attractors have become the object of research interest. The bundle of solutions introduced in this report can be regarded as a special case of strange attractors. In fact, we can consider Eqs. (3) as a special type of three-dimensional autonomous system in which all points of the form $(t + 2n\pi, x, y)$ $(n \in Z)$ are coincident. Therefore, it seems that a large part of our results can be applied to these problems.

5. CONCLUSION

Following our previous work,[12] randomly transitional processes have been studied in the physical system governed by Duffing's equation. By using analog and digital computers, experimental results are obtained about the dependence of the processes on the system parameters. From these results, the transition of the attractors and the average power spectra of the processes under the variation of the system parameters have been clarified.

The randomly transitional process develops from ultrasubharmonic oscillations through SI branching. This transition is reversible, while catastrophe through a transition chain is irreversible from the randomly transitional process to the periodic oscillation. The attractors representing the randomly transitional processes are characterized by exponentlike quantities. Time evolution characteristics of randomly transitional processes are represented by power spectra. The results obtained in the series of our reports will be applied to various physical problems. They also deserve attention as material for mathematical study.

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